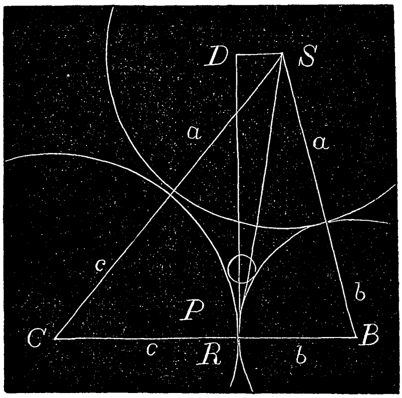
SOLUTION OF A PROBLEM.

BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.

THREE circles, radii a, b, c, touch each other externally; required the radii, a_1, b_1, c_1 of three circles described in the space enclosed by them, each touching the other two and two of the given circles.

A solution of this problem can be easily obtained by means of the equation proved at page 24, Vol. II. of the ANALYST.

$T^2 = 8aa_1$, T being the direct common tangent of the circles whose radii are a, a_1 and centers S, S'; this being a pair that do not touch each other.



Since the circles S, S', touch B, C, externally, their center of similitude P, which is the intersection of their direct common tangents, will be in the radical axis DR. If N, N', be the points where this common tangent, which is not shown in the figure, touches S, S', it is well known that $PR^2 = PN.PN'$. Again, $\frac{a}{a_1} = \frac{PN}{PN'}$, $\frac{a-a_1}{a_1} = \frac{PN-PN'}{PN'} = \frac{T}{PN'}$ $= \frac{2\sqrt{(2aa_1)}}{PN'}$, $\therefore PN' = \frac{2a_1\sqrt{(2aa_1)}}{a-a_1}$.

Similarly $\quad PN = \frac{2a\sqrt{(2aa_1)}}{a-a_1}$. $\quad \therefore PR = \frac{2aa_1\sqrt{2}}{a-a_1}$.

Draw SD perpendicular to the radical axis DR; then DR being the altitude of the triangle CBS, its value is $\sqrt{[abc(a+b+c)]} \div \frac{1}{2}(b+c)$.

Also $(a+b)^2 - (b-SD)^2 = DR^2 = (a+c)^2 - (c+SD)^2$

$$\therefore SD = a(c-b) \div (c+b)$$

Again $\sqrt{(PN^2 + a^2 - SD^2)} = PD = DR - PR$; whence

$$\sqrt{\left[\frac{8a^3a_1}{(a-a_1)^2} + a^2 - \frac{a^2(c-b)^2}{(c+b)^2}\right]} = \frac{2\sqrt{[abc(a+b+c)]}}{b+c} - \frac{2aa_1\sqrt{2}}{a-a_1},$$

an equation of the second degree in a_1.

Squaring this equation, reducing, and solving with reference to a_1,

$$a_1 = \frac{a\left\{bc+ac+ab+\sqrt{[2abc(a+b+c)]}\right\} \pm a^2(b+c) \pm a\sqrt{[2abc(a+b+c)]}}{bc+2ab+2ac+2\sqrt{[2abc(a+b+c)]}}$$

whence $a_1 = a$ or $\frac{1}{1 \div m - 1 \div a}$, where $\frac{1}{m} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} + 2\left(\frac{2}{ab} + \frac{2}{bc} + \frac{2}{ca}\right)^{\frac{1}{2}}$

Similarly $b_1 = \frac{1}{1 \div m - 1 \div b}$, $c_1 = \frac{1}{1 \div m - 1 \div c}$, the radii required.

The first value of a_1 is worthy of note; for if it were required to determine the position of four circles b, c, b_1, c_1 of given magnitude, that touch each other consecutively, so that a circle could be drawn touching all, the equation $a_1 = a$, shows that each of these circles fulfills this condition.

Again, a, b, c, being the radii of the first, we have the following relations connecting the radii of the circles of the various groups.

$$\left.\begin{array}{l}\frac{1}{a_1}=\frac{1}{m}-\frac{1}{a}\\ \frac{1}{b_1}=\frac{1}{m}-\frac{1}{b}\\ \frac{1}{c_1}=\frac{1}{m}-\frac{1}{c}\end{array}\right\}\text{2nd},\quad \left.\begin{array}{l}\frac{1}{a_2}=\frac{1}{m_1}-\frac{1}{a_1}\\ \frac{1}{b_2}=\frac{1}{m_1}-\frac{1}{b_1}\\ \frac{1}{c_2}=\frac{1}{m_1}-\frac{1}{c_1}\end{array}\right\}\text{3rd},\quad \left.\begin{array}{l}\frac{1}{a_x}=\frac{1}{m_{x-1}}-\frac{1}{a_{x-1}}\\ \frac{1}{b_x}=\frac{1}{m_{x-1}}-\frac{1}{b_{x-1}}\\ \frac{1}{c_x}=\frac{1}{m_{x-1}}-\frac{1}{c_{x-1}}\end{array}\right\}x+1.$$

Now substituting for $1 \div a_1$, $1 \div b_1$, $1 \div c_1$ in the third group, their values as given in the second, and carrying the resulting values of $1 \div a_2$, $1 \div b_2$, $1 \div c_2$ into the fourth, and so on, to the $(x+1)$th group, we find that

$$\frac{1}{a_x}+\frac{1}{a}=\frac{1}{b_x}+\frac{1}{b}=\frac{1}{c_x}+\frac{1}{c}=\frac{1}{m_{x-1}}-\frac{1}{m_{x-2}}+\ldots+\frac{1}{m},$$

when x is odd, and

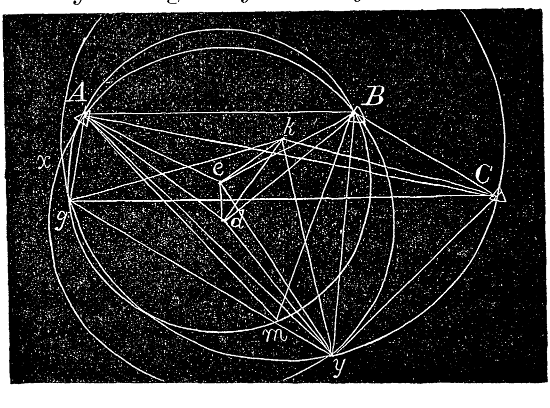
$$\frac{1}{a_x}-\frac{1}{a}=\frac{1}{b_x}-\frac{1}{b}=\frac{1}{c_x}-\frac{1}{c}=\frac{1}{m_{x-1}}-\frac{1}{m_{x-2}}+\ldots-\frac{1}{m}$$

when x is an even number.

A PROBLEM IN SURVEYING.

BY T. J. LOWRY, M. S., SAN FRANCISCO, CALIFORNIA.

Problem:—Required the positions of the two places of observation y and m, with reference to three known points A, B and C, having observed at m the angles AmB and Bmy and at y the angles ByA and Cym.



Trig. Analysis:—In the isosceles △ ABe we have the base AB and ∠ AeB ($=2AmB$), and hence all the ∠s to find Ae or Be. And in △ Ade are known ∠ Aed ($=180° - AmB$), side Ae, and ∠ Ade ($=AyB$) to get Ad and de. Now in isosceles △ Age having sides Ae and ge, and ∠ Aeg [=